7. curl (grad $\phi$) $= \vec{\nabla} \times (\vec{\nabla}\phi) = 0$

Pf. grad $\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

$\therefore$ curl (grad $\phi$) $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$

$= \left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right)\hat{i} + \left(\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right)\hat{j} + \left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)\hat{k}$

$= 0$

Note: This result means that (grad $\phi$) is always an irrotational vector.

8. $\operatorname{div}(\operatorname{curl}\vec{F}) = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$

Pf. Let $\vec{F} = F_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\therefore \operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{k}$$

$$\therefore \operatorname{div}(\operatorname{curl}\vec{F}) = \sum \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) = 0$$

Note: This result means that curl $\vec{F}$ is always a solenoidal vector.

9. $\operatorname{curl}(\operatorname{curl}\vec{F}) = \vec{\nabla}\times(\vec{\nabla}\times\vec{F}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{F}) - \vec{\nabla}^2\vec{F}$

Pf. Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\operatorname{curl}\vec{F} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\hat{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right.$$

$\therefore \operatorname{curl}(\operatorname{curl}\vec{F})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) & \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) & \left(\frac{\partial f_4}{\partial x} - \frac{\partial f_1}{\partial y}\right) \end{vmatrix}$$

$$= \sum \left[\frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\right]\hat{i}$$

$$= \sum \left[ \left( \frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} \right) - \left( \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right] \hat{i}$$

$$= \sum \left[ \left( \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_2}{\partial x \partial y} + \frac{\partial^2 F_3}{\partial x \partial z} \right) - \left( \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right] \hat{i}$$

$$= \sum \left[ \frac{\partial}{\partial x} \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) - \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{F_1} \right] \hat{i}$$

$$= \sum \left[ \frac{\partial}{\partial x} (\vec{\nabla} \cdot \vec{F}) - \vec{\nabla}^2 F_1 \right] \hat{i}$$

$$= \left[ i \frac{\partial}{\partial x} (\vec{\nabla} \cdot \vec{F}) + \hat{j} \frac{\partial}{\partial y} (\vec{\nabla} \cdot \vec{F}) + \hat{k} \frac{\partial}{\partial z} (\vec{\nabla} \cdot \vec{F}) \right] - \vec{\nabla}^2 (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k})$$

$$= \underline{\vec{\nabla} (\vec{\nabla} \cdot \vec{F}) - \vec{\nabla}^2 \vec{F}}$$

Which again implies

$$\text{grad}\,(\text{div}\, \vec{F}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{F}) = \vec{\nabla} \times (\vec{\nabla} \times \vec{F}) + \vec{\nabla}^2 \vec{F},$$

5. S.T. $\vec{F} = (y^2 - z^2 + 3yz - 2x)\hat{i} + (3xz + 2xy)\hat{j} + (3xy - 2xz + 2z)\hat{k}$ is both solenoidal & irrotational.

Soln. $\vec{\nabla}\cdot\vec{F} = \sum \frac{\partial}{\partial x}(y^2 - z^2 + 3yz - 2x) = -2 + 2x - 2x + 2$

$= 0 \quad \forall (x, y, z)$

$\therefore$ $\vec{F}$ is solenoidal.

$\vec{\nabla}\times\vec{F} = (3x - 3x)\hat{i} - (3y - 2z + 2z - 3y)\hat{j} + (3z + 2y - 2y - 3z)\hat{k}$

$= 0, \quad \forall (x, y, z)$

$\therefore$ $\vec{F}$ is irrotational.

6. S.T. $\vec{F} = (y^2 + 2xz^2)\hat{i} + (2xy - z)\hat{j} + (2x^2z - y + 2z)\hat{k}$ is irrotational and hence find its scalar potential.

Soln. $\vec{\nabla}\times\vec{F} = 0 \quad \forall (x, y, z)$.

$\therefore$ $\vec{F}$ is irrotational.

Let the scalar potential of $\vec{F}$ by $\phi$.

$\therefore$ $\vec{F} = \vec{\nabla}\phi = \sum \frac{\partial \phi}{\partial x}\hat{i}$

$\therefore$ $\frac{\partial \phi}{\partial x} = y^2 + 2xz^2$ (int. partially w.r.t. $x$)

$\Rightarrow \phi = xy^2 + x^2z^2 +$ a function indep. of $x \longrightarrow$ ①

$\frac{\partial \phi}{\partial y} = 2xy - z \Rightarrow \phi = xy^2 - yz +$ a function indep. of $y$ $\longrightarrow$ ②

$\frac{\partial \phi}{\partial z} = 2x^2z - y + 2z \Rightarrow \phi = x^2z^2 - yz + z^2 +$ a fun. indep of $z \longrightarrow$ ③

From (1), (2) & (3), we get $\phi = xy^2 + x^2z^2 - yz + z^2 + c$

#7. Find the values of constants a, b, c so that $\vec{F} = (axy + bz^3)\hat{i} + (3x^2 - cz)\hat{j} + (3xz^2 - y)\hat{k}$ may be irrotational. For these values of ~~a~~ a, b, c find also the scalar potential of $\vec{F}$.

Soln: $\vec{F}$ is irrotational $\Rightarrow \vec{\nabla} \times \vec{F} = 0$

$$\Rightarrow (-1 + c)\hat{i} - (3z^2 - 3bz^2)\hat{j} + (6x - ax)\hat{k} = 0$$

$$\Rightarrow c - 1 = 0, \quad 3z^2(1 - b) = 0, \quad x(6 - a) = 0$$

or $c = 1, \ b = 1, \ a = 6$.

Using these values of a, b, c,

$$\vec{F} = (6xy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (3xz^2 - y)\hat{k}.$$

If $\phi$ be the scalar potential of $\vec{F}$, $\vec{F} = \vec{\nabla}\phi$

$$\therefore \vec{F} = \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$$

$$\therefore \frac{\partial \phi}{\partial x} = 6xy + z^3, \quad \frac{\partial \phi}{\partial y} = 3x^2 - z, \quad \frac{\partial \phi}{\partial z} = 3xz^2 - y.$$

Integrating partially w.r.t. the concerned variables

$\phi = 3x^2y + xz^3 +$ a funct indep of $x$

$\phi = 3x^2y - yz +$ a funct indep of $y$

$\phi = xz^3 - yz +$ a funct indep of $z$.

$\therefore$ we get $\phi = 3x^2y + xz^3 - yz + c$

13. If $r = |\vec{r}|$, where $\vec{r}$ is the p.v. of $(x, y, z)$ w.r.t. origin, P.T. i) $\vec{\nabla} f(r) = \frac{f'(r)}{r}\vec{r}$ and

ii) $\vec{\nabla}^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Solⁿ. i) $r^2 = x^2+y^2+z^2$, $\vec{r} = \sum x\hat{i}$

$2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$. Similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\vec{\nabla} f(r) = \sum \hat{i}\frac{\partial}{\partial x} f(r) = \sum \hat{i}\frac{\partial}{\partial r} f(r)\frac{\partial r}{\partial x}$$

$$= f'(r)\sum \hat{i}\frac{\partial r}{\partial x} = f'(r)\sum \hat{i}\frac{x}{r} = \frac{f'(r)}{r}\vec{r}.$$

ii) $\vec{\nabla}^2 f(r) = \vec{\nabla}\cdot\vec{\nabla} f(r) = \vec{\nabla}\cdot\left(\frac{f'(r)}{r}\vec{r}\right)$

$= \vec{\nabla}\left(\frac{f'(r)}{r}\right)\cdot\vec{r} + \frac{f'(r)}{r}(\vec{\nabla}\cdot\vec{r})$

$$\boxed{\begin{aligned} &v = f(u)\\ &\vec{\nabla} v = f'(u)\vec{\nabla} u\\ \hline &\vec{\nabla}(r)\\ &= \sum \frac{\partial r}{\partial x}\hat{i}\\ &= \sum \frac{x}{r}\hat{i}\\ &= \frac{1}{r}\vec{r}\end{aligned}}$$

$= \left[\frac{rf''(r) - f'(r)}{r^2}\right]\vec{\nabla}(r)\cdot\vec{r} + 3\frac{f'(r)}{r}$

$= \frac{rf''(r) - f'(r)}{r^2}\left(\frac{1}{r}\vec{r}\right)\cdot\vec{r} + \frac{3f'(r)}{r}$

$= \frac{rf''(r) - f'(r)}{r^2}\frac{1}{r}(r^2) + \frac{3f'(r)}{r}$

$= f''(r) + \frac{2}{r}f'(r)$

14. Find $f(r)$, if the vector $f(r)\vec{r}$ is both solenoidal and irrotational.

Soln. As $f(r)\vec{r}$ is solenoidal $\Rightarrow \vec{\nabla}\cdot[f(r)\vec{r}] = 0$

i.e., $\vec{\nabla} f(r)\cdot\vec{r} + f(r)\vec{\nabla}\cdot\vec{r} = 0$ (by expansion formulas)

i.e., $\left(\frac{f'(r)}{r}\vec{r}\right)\cdot\vec{r} + 3f(r) = 0$ ($\because \vec{\nabla}\cdot\vec{r} = 3$ & by previous prob)

i.e., $\frac{f'(r)}{f(r)} + \frac{3}{r} = 0$ ($\because \vec{r}\cdot\vec{r} = r^2$)

Int. both sides w.r.t. $r$

$\log f(r) + 3\log r = \log c$

or $\log(r^3 f(r)) = \log c$

$\Rightarrow f(r) = \frac{c}{r^3}$ $\longrightarrow$ (i)

Again $f(r)\vec{r}$ is also irrotational.

$\Rightarrow \vec{\nabla}\times[f(r)\vec{r}] = 0$

i.e., $\vec{\nabla} f(r)\times\vec{r} + f(r)\,\vec{\nabla}\times\vec{r} = 0$ (by e.f.)

or $\left(\frac{f'(r)}{r}\vec{r}\right)\times\vec{r} + f(r).0 = 0$ (as $\vec{\nabla}\times\vec{r} = 0$)

or $\frac{f'(r)}{r} \cdot 0 + 0 = 0$, which is true for all values of $f(r)$. $\longrightarrow$ (2)

from (1) & (2) we get $f(r)\vec{r}$ is both solenoidal and irrotational if $f(r) = \frac{c}{r^3}$.

# LINE INTEGRAL OF VECTOR POINT FUNCTIONS

Let $\overline{F}(x, y, z)$ be a vector point function defined at all points in some region of space and let $C$ be a curve in that region.

Let the position vectors of two neighbouring points $P$ and $Q$ on $C$ be $\overline{r}$ and $\overline{r} + \Delta \overline{r}$. Then $\overline{PQ} = \Delta \overline{r}$. If $\overline{F}$ acts at $P$ in a direction that makes an angle $\theta$ with $\overline{PQ}$,

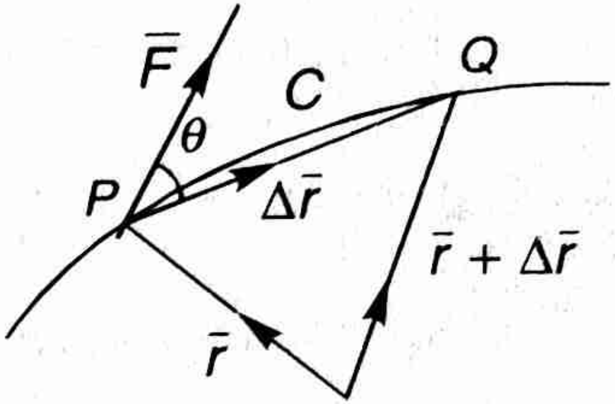

Fig. 2.2

then $\overline{F} \cdot \Delta\overline{r} = F(\Delta r)\cos\theta$

In the limit, $\overline{F} \cdot \mathrm{d}\overline{r} = F\,\mathrm{d}r\cos\theta$.

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**Note** Physically $\overline{F} \cdot \mathrm{d}\overline{r}$ means the elemental work done by the force $\overline{F}$ through the displacement $\mathrm{d}\overline{r}$.

Now the integral $\int_C \overline{F} \cdot \mathrm{d}\overline{r}$ is defined as *the line integral of $\overline{F}$ along the curve C.*

Since $\int_C \overline{F} \cdot \mathrm{d}\overline{r} = \int_C F\cos\theta\,\mathrm{d}r$, it is also called the line integral of the tangential component of $\overline{F}$ along $C$.

---

**Note** (1) $\int_{A\,(C)}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ depends not only on the curve $C$ but also on the terminal points $A$ and $B$.

(2) Physically $\int_{A\,(C)}^{B} \overline{F} \cdot \mathrm{d}\overline{r}$ denotes the total work done by the force $\overline{F}$ in displacing a particle from $A$ to $B$ along the curve $C$.

(3) If the value of $\int_A^B \overline{F} \cdot \mathrm{d}\overline{r}$ does not depend on the curve $C$, but only on the terminal points $A$ and $B$, $\overline{F}$ is called a *Conservative vector*. Similarly, if the work done by a force $\overline{F}$ in displacing a particle from $A$ to $B$ does not depend on the curve along which the particle gets displaced but only on $A$ and $B$, the force $\overline{F}$ is called a *Conservative force*.

(4) If the path of integration $C$ is a closed curve, the line integral is denoted as $\oint_C \overline{F} \cdot \mathrm{d}\overline{r}$.

(5) When $\overline{F} = F_1\overline{i} + F_2\,\overline{j} + F_3\,\overline{k}$,

$$\int_C \overline{F} \cdot \mathrm{d}\overline{r} = \int_C (F_1\overline{i} + F_2\,\overline{j} + F_3\,\overline{k}) \cdot (\mathrm{d}x\overline{i} + \mathrm{d}y\overline{j} + \mathrm{d}z\overline{k})$$

$$(\because \overline{r} = x\overline{i} + y\overline{j} + z\overline{k})$$

$$= \int_C (F_1\mathrm{d}x + F_2\mathrm{d}y + F_3\mathrm{d}z),$$ which is evaluated as in the problems in Chapter 1.

(6) $= \int_C \phi\,\mathrm{d}r$, where $\phi$ is a scalar point function and $\int_C \overline{F} \times \mathrm{d}\overline{r}$ are also line integrals.

## Condition for $\overline{F}$ to be Conservative

If $\overline{F}$ is an irrotational vector, it is conservative.

***Proof:*** Since $\overline{F}$ is irrotational, it can be expressed as $\nabla\phi$, i.e., $\overline{F} = \nabla\phi$

$$\int_A^B \overline{F}\cdot d\overline{r} = \int_A^B \nabla\phi\cdot d\overline{r}$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}\overline{i} + \frac{\partial\phi}{\partial y}\overline{j} + \frac{\partial\phi}{\partial z}\overline{k}\right)\cdot(dx\overline{i} + dy\overline{j} + dz\overline{k})$$

$$= \int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$= \int_A^B d\phi$$

$= [\phi]_A^B$, whatever be the path of integration

$$= \phi(B) - \phi(A)$$

$\therefore$ $\overline{F}$ is conservative.

**Note** If $\overline{F}$ is irrotational (and hence conservative) and $C$ is a closed curve, then $\oint_C \overline{F}\cdot d\overline{r} = 0$.

$[\because \phi(A) = \phi(B)$, as $A$ and $B$ coincide$]$

## Surface Integral of Vector Point Function

Let $S$ be a two sided surface, one side of which is considered arbitrarily as the positive side.

Let $\overline{F}$ be a vector point function defined at all points of $S$.

Let $dS$ be the typical elemental surface area in $S$ surrounding the point $(x, y, z)$.

Let $\hat{n}$ be the unit vector normal to the surface $S$ at $(x, y, z)$ drawn in the positive side (or outward direction)

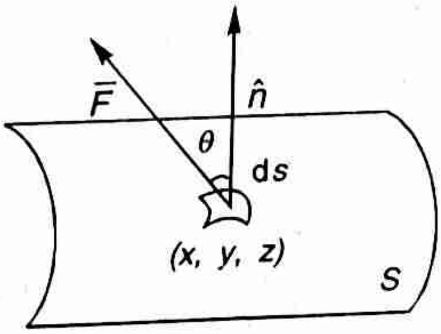

**Fig. 2.3**

Let $\theta$ be the angle between $\overline{F}$ and $\hat{n}$.

$\therefore$ The normal component of $\overline{F} = \overline{F}\cdot\hat{n} = F\cos\theta$

The integral of this normal component through the elemental surface area $dS$ over the surface $S$ is called *the surface integral of* $\overline{F}$ *over* $S$ and denoted as $\int_S F\cos\theta\, dS$ or $\int_S \overline{F}\cdot\hat{n}\, dS$.

If $d\overline{S}$ is a vector whose magnitude is $dS$ and whose direction is that of $\hat{n}$, then $d\overline{S} = \hat{n}\, dS$.

$\therefore \int_S \overline{F}\cdot\hat{n}\, dS$ can also be written as $\int_S \overline{F}\cdot d\overline{S}$.

**Note** (1) If $S$ is a closed surface, the outer surface is usually chosen as the positive side.

(2) $\int\limits_S \phi \, \mathrm{d}\bar{S}$ and $\int\limits_S \bar{F} \times \mathrm{d}\bar{S}$, where $\phi$ is a scalar point function are also surface integrals.

(3) When evaluating $\int\limits_S \bar{F} \cdot \hat{n} \, \mathrm{d}\bar{S}$, the surface integral is first expressed in the scalar form and then evaluated as in problems in Chapter 1.

To evaluate a surface integral in the scalar form, we convert it into a double integral and then evaluate. Hence the surface integral $\int\limits_S \bar{F} \cdot \mathrm{d}\bar{S}$ is also denoted as $\iint\limits_S \bar{F} \cdot \mathrm{d}\bar{S}$.

## Worked Example 2(c)

**Example 1** Evaluate $\int\limits_C \phi \, \mathrm{d}\bar{r}$, where $C$ is the curve $x = t$, $y = t^2$, $z = (1 - t)$ and $\phi = x^2 y (1 + z)$ from $t = 0$ to $t = 1$.

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

$$\therefore \qquad \mathrm{d}\bar{r} = \mathrm{d}x\bar{i} + \mathrm{d}y\bar{j} + \mathrm{d}z\bar{k}$$

Hence the given line integral $\mathrm{I} = \int\limits_C x^2 y (1 + z) (\mathrm{d}x\bar{i} + \mathrm{d}y\bar{j} + \mathrm{d}z\bar{k})$

$$= \bar{i} \int\limits_C x^2 y (1 + z) \, \mathrm{d}x + \bar{j} \int\limits_C x^2 y (1 + z) \, \mathrm{d}y + \bar{k} \int\limits_C x^2 y (1 + z) \mathrm{d}z$$

$$= \bar{i} \int_0^1 t^4 (2 - t) \, \mathrm{d}t + \bar{j} \int_0^1 t^4 (2 - t) \, 2t \, \mathrm{d}t + \bar{k} \int_0^1 t^4 (2 - t) (-\mathrm{d}t)$$

$$= \bar{i} \left[ 2\frac{t^5}{5} - \frac{t^6}{6} \right]_0^1 + \bar{j} \left[ 4\frac{t^6}{6} - 2\frac{t^7}{7} \right]_0^1 + \bar{k} \left[ -2\frac{t^5}{5} + \frac{t^6}{6} \right]_0^1$$

$$= \frac{7}{30}\bar{i} + \frac{8}{21}\bar{j} - \frac{7}{30}\bar{k}.$$

**Example 2** If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$, evaluate $\int\limits_C \bar{F} \times \mathrm{d}\bar{r}$, where $C$ is the curve $x = t^2$, $y = 2t$, $z = t^3$ from (0, 0, 0 ) to (1, 2, 1).

$$\bar{F} \times \mathrm{d}\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ xy & -z & x^2 \\ \mathrm{d}x & \mathrm{d}y & \mathrm{d}z \end{vmatrix}$$

$$= -(z \, \mathrm{d}z + x^2 \, \mathrm{d}y)\bar{i} - (xy \, \mathrm{d}z - x^2 \, \mathrm{d}x)\bar{j} + (xy \, \mathrm{d}y + z\mathrm{d}x)\bar{k}$$

∴ The given line integral

$$=\int_C [-(z\,\mathrm{d}z + x^2\,\mathrm{d}y)\bar{i} - (xy\,\mathrm{d}z - x^2\,\mathrm{d}x)\bar{j} + (xy\,\mathrm{d}y + z\,\mathrm{d}x)\bar{k}]$$

$$=\int_0^1 [-(t^3 \cdot 3t^2 \cdot + t^4 \cdot 2)\mathrm{d}t\,\bar{i} - (2t^3 \cdot 3t^2 - t^4 \cdot 2t)\mathrm{d}t\,\bar{j} + (2t^3 \cdot 2 + t^3\,2\,t)\mathrm{d}t\,\bar{k}]$$

[∵ (0, 0, 0) corresponds to $t = 0$ and (1, 2, 1) corresponds to $t = 1$]

$$=-\bar{i}\int_0^1 (3t^5 + 2t^4)\,\mathrm{d}t - \bar{j}\int_0^1 (6t^5 - 2t^5)\mathrm{d}t + \bar{k}\int_0^1 (4t^3 + 2t^4)\mathrm{d}t$$

$$=-\bar{i}\left(3\frac{t^6}{6} + 2\frac{t^5}{5}\right)_0^1 - \bar{j}\left(4\frac{t^6}{6}\right)_0^1 + \bar{k}\left(t^4 + 2\frac{t^5}{5}\right)_0^1$$

$$=-\frac{9}{10}\bar{i} - \frac{2}{3}\bar{j} + \frac{7}{5}\bar{k}.$$

**Example 3** Find the work done when a force $\overline{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$ displaces a particle in the $xy$-plane from (0, 0) to (1, 1) along the curve (i) $y = x$, (ii) $y^2 = x$. Comment on the answer.

$W$ = Work done by $\overline{F} = \int_C \overline{F} \cdot \mathrm{d}\bar{r}$

$$=\int_C [(x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}] \cdot (\mathrm{d}x\bar{i} + \mathrm{d}y\bar{j} + \mathrm{d}z\bar{k})$$

$$=\int_C [(x^2 - y^2 + x)\,\mathrm{d}x - (2xy + y)\mathrm{d}y]$$

***Case (i)*** $C$ is the line $y = x$.

$$\therefore \quad W_1 = \int_{\substack{y = x \\ (\mathrm{d}y = \mathrm{d}x)}} [(x^2 - y^2 + x)\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

$$= \int_0^1 (-2x^2)\mathrm{d}x$$

$$= -\frac{2}{3}$$

***Case (ii):*** $C$ is the curve $y^2 = x$.

$$\therefore \quad W_2 = \int_{\substack{x = y^2 \\ (\mathrm{d}x = 2y\,\mathrm{d}y)}} [(x^2 - y^2 + x)\mathrm{d}x - (2xy + y)\,\mathrm{d}y]$$

$$= \int_0^1 (2y^5 - 2y^3 - y)\,\mathrm{d}y$$

$$= -\frac{2}{3}$$

***Comment*** As the works done by the force, when it moves the particle along two different paths from (0, 0) to (1, 1), are equal, the force may be a conservative force. In fact, $\overline{F}$ is a conservative force, as $\overline{F}$ is irrotational.

It can be verified that the work done by $\overline{F}$ when it moves the particle from (0, 0) to (1, 1) along any other path (such as $x^2 = y$ ) is also equal to $-\frac{2}{3}$.

**Example 4** Find the work done by the force $\overline{F} = z\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r} = \cos t\bar{i} + \sin t\bar{j} + t\bar{k}$ from $t = 0$ to $t = 2\pi$.

From the vector equation of the curve $C$, we get the parametric equations of the curve as $x = \cos t$, $y = \sin t$, $z = t$.

Work done by $\overline{F}$

$$= \int_C \overline{F} \cdot \mathrm{d}\bar{r}$$

$$= \int_C (z\bar{i} + x\bar{j} + y\bar{k}) \cdot (\mathrm{d}x\bar{i} + \mathrm{d}y\bar{j} + \mathrm{d}z\bar{k})$$

$$= \int_C (z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z)$$

$$= \int_0^{2\pi} [t(-\sin t) + \cos^2 t + \sin t]\mathrm{d}t$$

$$= \left[t\cos t - \sin t + \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right) - \cos t\right]_0^{2\pi}$$

$$= (2\pi + \pi - 1) - (-1)$$

$$= 3\pi$$

**Example 5** Evaluate $\oint_C \overline{F} \cdot \mathrm{d}\bar{r}$, where $\overline{F} = (\sin y)\bar{i} + x(1 + \cos y)\bar{j} + z\bar{k}$ and $C$ is the circle $x^2 + y^2 = a^2$ in the $xy$-plane.

Given integral

$$= \int_C [\sin y\bar{i} + x(1 + \cos y)\bar{j} + z\bar{k}] \cdot (\mathrm{d}x\bar{i} + \mathrm{d}y\bar{j} + \mathrm{d}z\bar{k})$$

$$= \int_{\left(\substack{x^2 + y^2 = a^2 \\ z = 0}\right)} [\sin y\,\mathrm{d}x + x(1 + \cos y)\,\mathrm{d}y + z\,\mathrm{d}z]$$

$$= \int_{x^2 + y^2 = a^2} [\sin y\,\mathrm{d}x + x(1 + \cos y)\mathrm{d}y]$$

Since $C$ is a closed curve, we use the parametric equations of $C$, namely $x = a\cos\theta$, $y = a\sin\theta$ and the parameter $\theta$ as the variable of integration. To move around the circle $C$ once completely, $\theta$ varies from 0 to $2\pi$.

Now, given integral

$$= \int [(\sin y\,\mathrm{d}x + x\cos y\,\mathrm{d}y) + x\,\mathrm{d}y]$$

$$= \int [\mathrm{d}(x\sin y) + x\,\mathrm{d}y]$$

$$= \int_0^{2\pi} \{\mathrm{d}\,[a\cos\theta \cdot \sin(a\sin\theta)] + a^2\cos^2\theta\,\mathrm{d}\theta\}$$

$$= \left[ a\cos\theta \cdot \sin(a\sin\theta) + \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_0^{2\pi}$$

$$= \pi a^2$$

**Example 6** Find the work done by the force $\bar{F} = (2xy + z^3)\bar{i} + x^2\,\bar{j} + 3xz^2\,\bar{k}$, when it moves a particle from (1, –2, 1) to (3, 1, 4) along any path.

To evaluate the work done by a force, the equation of the path and the terminal points must be given. As the equation of the path is not given in this problem, we guess that the given force $\bar{F}$ is conservative. Let us verify whether $\bar{F}$ is conservative, i. e. irrotational.

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= (0 - 0)\bar{i} - (3z^2 - 3z^2)\bar{j} + (2x - 2x)\bar{k}$$

$$= 0$$

$\therefore$ $\bar{F}$ is irrotational and hence conservative.

$\therefore$ Work done by $\bar{F}$ depends only on the terminal points.

Since $\bar{F}$ is irrotational, let $\bar{F} = \nabla\phi$.

It is easily found that $\phi = x^2y + z^3x + c$.

$$\text{Work done by } \bar{F} = \int_{(1,-2,1)}^{(3,1,4)} \bar{F} \cdot d\bar{r}$$

$$= \int_{(1,-2,1)}^{(3,1,4)} \nabla\phi \cdot d\bar{r}$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\phi$$

$$= [\phi(x, y, z)]_{(1,-2,1)}^{(3,1,4)}$$

$$= \left[x^2y + z^2x + c\right]_{(1,-2,1)}^{(3,1,4)}$$

$$= (201 + c) - (-1 + c)$$

$$= 202.$$

**Example 7** Find the work done by the force $\bar{F} = y(3x^2y - z^2)\bar{i} + x(2x^2y - z^2)\bar{j} - 2xyz\bar{k}$, when it moves a particle around a closed curve $C$. To evaluate the work done by a force, the equation of the path $C$ and the terminal points must be given.

Since $C$ is a closed curve and the particle moves around this curve once completely, any point $(x_0, y_0, z_0)$ can be taken as the initial as well as the final point.

But the equation of $C$ is not given. Hence we guess that the given force $\overline{F}$ is conservative, i.e. irrotational. Actually it is so, as verified below.

$$\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3y^2x^2 - yz^2 & 2x^3y - z^2x & -2xyz \end{vmatrix}$$

$$= (-2xz + 2xz)\overline{i} - (-2yz + 2yz)\overline{j} + (6x^2y - z^2 - 6x^2y + z^2)\overline{k}$$

$$= 0$$

Since $\overline{F}$ is irrotational, let $\overline{F} = \nabla\phi$.

$\therefore$ Work done by $\overline{F} = \oint\limits_C \overline{F} \cdot \mathrm{d}\overline{r}$

$$= \oint\limits_C \nabla\phi \cdot \mathrm{d}\overline{r}$$

$$= \int_{(x_0, y_0, z_0)}^{(x_0, y_0, z_0)} \mathrm{d}\phi$$

$$= \phi(x_0, y_0, z_0) - \phi(x_0, y_0, z_0)$$

$$= 0$$

**Example 8** Evaluate $\iint\limits_S \overline{A} \cdot \mathrm{d}\overline{S}$, where $\overline{A} = 12x^2y\overline{i} - 3yz\overline{j} + 2z\overline{k}$ and $S$ is the portion of the plane $x + y + z = 1$ included in the first octant.

Given integral $\mathrm{I} = \iint\limits_S \overline{A} \cdot \hat{n}\, \mathrm{d}\overline{S}$, where $\hat{n}$ is the unit normal to the surface $S$ given by

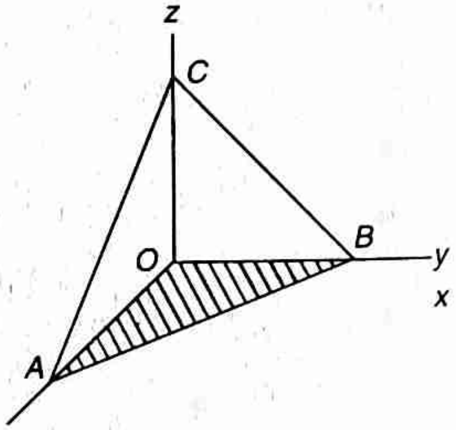

Fig. 2.4

$\phi = c$,

i.e. $x + y + z = 1$

$\therefore$ $\phi = x + y + z$

$$\nabla\phi = \overline{i} + \overline{j} + \overline{k}$$

$$\hat{n} = \frac{1}{\sqrt{3}}(\overline{i} + \overline{j} + \overline{k})$$

$\therefore$ $$\mathrm{I} = \iint\limits_S (12x^2y\overline{i} - 3yz\overline{j} + 2z\overline{k}) \cdot \frac{1}{\sqrt{3}}(\overline{i} + \overline{j} + \overline{k})\, \mathrm{d}S$$

$$= \frac{1}{\sqrt{3}} \iint\limits_S (12x^2y - 3yz + 2z)\, \mathrm{d}S$$

To convert the surface integral as a double integral, we project the surface $S$ on the $xoy$ - plane. Then $dS \cos \gamma = dA$, where $\gamma$ is the angle between the surface $S$ and the $xoy$ - plane, i.e. the angle between $\hat{n}$ and $\bar{k}$. $\therefore \cos \gamma = \hat{n} \cdot \bar{k}$

$$\therefore \quad dS = \frac{dA}{\hat{n} \cdot \bar{k}} = \frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

$$\therefore \quad I = \frac{1}{\sqrt{3}} \iint_{\Delta OAB} (12x^2 y - 3yz + 2z) \frac{dx\,dy}{\frac{1}{\sqrt{3}}}$$

[$\because$ the projection of $S$ on the xoy plane is $\Delta OAB$]

$$= \iint_{\Delta OAB} \{12x^2 y - 3y(1 - x - y) + 2(1 - x - y)\}\, dx\,dy$$

**Note** To express the integrand as a function of $x$ and $y$ only, $z$ is expressed as a function of $x$ and $y$ from the equation of $S$.

$$I = \int_0^1 \int_0^{1-y} (12x^2 y + 3xy + 3y^2 - 5y - 2x + 2)\, dx\,dy$$

$$= \int_0^1 \left[ 4y(1-y)^3 + \frac{3y}{2}(1-y)^2 + 3y^2(1-y) - 5y(1-y) - (1-y)^2 + 2(1-y) \right] dy$$

$$= \frac{49}{120}$$

**Example 9** Evaluate $\iint_S \bar{F} \cdot d\bar{S}$, where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and $S$ is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies in the first octant.

Given integral $I = \iint_S \bar{F} \cdot \hat{n}\, dS$, where $\hat{n}$ is the unit normal to the surface $S$ given by

$$\phi = c \text{ i.e. } \quad x^2 + y^2 + z^2 = 1.$$

$$\phi = x^2 + y^2 + z^2$$

$$\therefore \quad \nabla \phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\therefore \quad \hat{n} = \frac{2x\bar{i} + 2y\bar{j} + 2z\bar{k}}{\sqrt{4(x^2 + y^2 + z^2)}}$$

$$= \frac{2(x\bar{i} + y\bar{j} + z\bar{k})}{\sqrt{4 \times 1}} \qquad [\because \text{ the point } (x, y, z) \text{ lies on } S]$$

$$= x\bar{i} + y\bar{j} + z\bar{k}.$$

$$\therefore \quad I = \iint_S (yz\bar{i} + zx\bar{j} + xy\bar{k}) \cdot (x\bar{i} + y\bar{j} + z\bar{k})\, dS$$

$$= \iint_S 3xyz \, dS$$

$$= \iint_R 3xyz \frac{dx\,dy}{\hat{n} \cdot \bar{k}},$$

where $R$ is the region in the $xy$-plane bounded by the circle $x^2 + y^2 = 1$ and lying in the first quadrant.

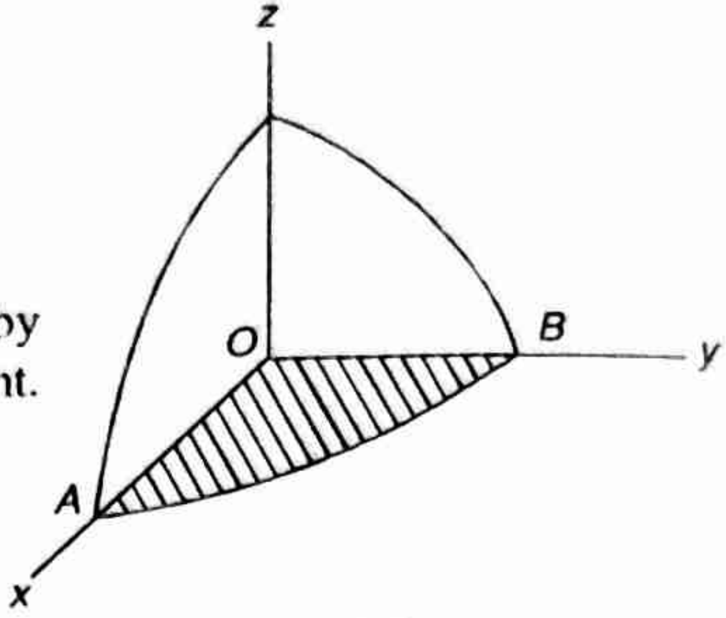

Fig. 2.5

$$I = \iint_R 3xyz \frac{dx\,dy}{z}$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} 3xy \, dx \, dy$$

$$= \int_0^1 3y \left( \frac{x^2}{2} \right)_0^{\sqrt{1-y^2}} dy$$

$$= \frac{3}{2} \int_0^1 y(1 - y^2) \, dy$$

$$= \frac{3}{2} \left( \frac{y^2}{2} - \frac{y^4}{4} \right)_0^1$$

$$= \frac{3}{8}$$